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Fractional Order Functional Integro-Differential Equation in Banach Algebras

B.D. Karande

Department of Mathematics,
Maharashtra Udayagiri Mahavidyalaya,
Udgir-413517, Maharashtra, India

E-mail: bdkarande@rediffmail.com

ABSTRACT

In this paper , we prove existence of locally attractive solution for the fractional order functional integro-differential equation in Banach Algebras are proved under mixed generalized lipschitz,Caratheodory and monotonicity conditions. By using a fixed point theorem for multi-valued maps due to Dhage, a main existence theorem is established.

Keywords: Banach algebra, functional integro-differential equation, existence result.

1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject has its origin in the 1600s. During three Centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. In the last few decades, however, fractional differentiation proved very useful in various fields of applied sciences and engineering. The class of fractional operator equations of various types plays very important role not only in mathematics but also in physics, chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrochemistry, viscoelasticity, feedback amplifier and electrical circuits. The 'bible 'of fractional calculus is the book of Samko, Kilbas and Marichev (1993). Recently fractional differential equations attracted many authors (see references in these papers). Using a fixed point theorem, Dhage's hybrid fixed point theorem, several results of existence have been obtained in the literature to linear and nonlinear equations and recently also to fractional differential equations.

Let $\mathbb{R}$ denote the real line, let $I_0 = [-q, 0]$ and $I = [0, T]$, $q, T \geq 0$, be two closed intervals in $\mathbb{R}_+$, and let $J = I_0 \cup I$. Let $C = C(I_0, \mathbb{R})$ be the space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|.\|_C$ defined by $\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|$.

Clearly, C is Banach algebra with this norm. we let $AC^1(J, \mathbb{R})$ be the space of real-valued continuous functions whose first derivatives exist and are absolutely continuous on J.

We consider the fractional order functional integro-differential equation (FIDE)

$$\left.\begin{aligned} &\frac{d^\alpha}{dt^\alpha}\left(\frac{x(t)}{f(t, x(t))}\right) = \int_0^t g(s, x_s)\,ds, \;\; a.e. \;\; t \in I, \\ &x(t) = \phi(t), \;\; t \in I_0, \end{aligned}\right\} \tag{1}$$

where d^α / dt^α denote the Riemann-Liouville derivative of order $\alpha, 0 < \alpha < 1$ and the function $x_t : I_0 \to C$ is the continuous function defined by $x_t(\theta) := x(t + \theta)$ for all $\theta \in I_0$ also $f : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : I \times C \to \mathbb{R}$, under suitable mixed Lipschitz and other conditions on the nonlinearities f and g.

By a solution of the FIDE (1.1), we mean a function $x \in AC^1(J, \mathbb{R})$ such that:

(i) the function $t \to \left(\dfrac{x}{f(t, x)}\right)$ is absolutely continuous for each $x \in \mathbb{R}$;

(ii) x Satisfies (1).

While fractional order functional differential equations have been a very active area of research for a long time, the study of fractional order functional integro-differential equations in Banach algebras is relatively new to the literature. The fixed point theorems of Dhage (2004) to be used in the following section.

2. AUXILIARY RESULTS

In this section we give the notations, definitions, hypotheses and preliminary tools, which will be used in the sequel.

Let $X = AC(\mathbb{R}_+, \mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and Ω be a subset of X. Let a mapping $A: X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (Ax)(t) \tag{2}$$

for all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for operator equation (2) on $\mathbb{R}_+$. We need the following definitions in the sequel.

Definition 2.1: (Mohammed (2010)). We say that solutions of the equation (2) are locally attractive if there exists a closed ball $\overline{B_r(x_0)}$ in the space $AC(\mathbb{R}_+, \mathbb{R})$ for some $x_0 \in AC(\mathbb{R}_+, \mathbb{R})$ and for some real number $r > 0$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2) belonging to $\overline{B_r(x_0)} \cap \Omega$ we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0 \tag{3}$$

Definition 2.2: Let X be a Banach space. A mapping $A: X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Ax - Ay\| \le \alpha \|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then A is called a contraction on X with the contraction constant α.

Definition 2.3: (Dugundji and Granas (1982)). An operator *A* on a Banach space X into itself is called Compact if for any bounded subset *S* of *X*, *A(S)* is a relatively compact subset of *X*. If A is continuous and compact, then it is called completely continuous on *X*.

Definition 2.4: Let X be a Banach space with the norm $\|.\|$ and let $A: X \to X$, be an operator (in general nonlinear). Then A is called

(i) Compact if $A(X)$ is relatively compact subset of X;
(ii) totally bounded if $A(S)$ is a totally bounded subset of X for any bounded subset S of X
(iii) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

We seek the solutions of (1) in the space $AC(J,\mathbb{R})$ of continuous and bounded real-valued functions defined on J. Define a standard supremum norm $\|.\|$ and a multiplication "." in $AC(J,\mathbb{R})$ by

$$\|x\| = \sup\{|x(t)| : t \in J\}, (xy)(t) = x(t)y(t), t \in J \tag{4}$$

Clear, $AC(J,\mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $L^1(J,\mathbb{R})$ we denote the space of Lebesgue integrable functions on J with the norm $\|.\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|dt \tag{5}$$

Definition 2.5. (Samko, Kilbas and Marivchev (1993)). Let $f \in L^1[0,T]$ and $\alpha > 0$. The Riemann-Liouville fractional derivative of order α α of real function f is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds \quad 0 < \alpha < 1$$

Such that $D^{-\alpha} f(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$ respectively.

For convenience, $D^{-\alpha}\{D^\alpha f(t)\} = f(t)$.

Definition 2.6. (Samko, Kilbas and Marivchev (1993)). The Riemann-Liouville fractional integral of order $\alpha \in (0,1)$ of the function $f \in L^1[0,T]$ is defined by the formula:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0,T] \tag{6}$$

where $I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$ $\Gamma(\alpha)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order α is defined by $D^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}} := \frac{d}{dt} \circ I^{1-\alpha}$.

It may be shown that the fractional integral operator I^{α} transforms the space $L^1(J,\mathbb{R})$ into itself and has some other properties.

Theorem 2.1: (Arzela-Ascoli theorem). If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(J,\mathbb{R})$, then it has a convergent subsequence.

Theorem 2.2: A metric space X is compact if every sequence in X has a convergent subsequence.

The nonlinear alternative of Schaefer type recently proved by Dhage (2004) is embodied in the following theorem.
Theorem 2.1: (Dhage (2004)). Let S be a non-empty, closed-convex and bounded subset of the Banach space X and let $A: X \to X$ and $B: S \to X$ are two operators satisfying

(a) A is Lipschitz with a Lipschitz constant α,
(b) B is completely continuous, and
(c) $AxBx \in S$ for all $x \in S$, and
(d) $\alpha M < 1$, where $M = \| B(S) \| := \sup\{ \; \| Bx \| \; : x \in S \}$.

Then operator the equation $AxBx = x$ has a solution in S.

3. EXISTENCE OF SOLUTIONS

Let $B(J,\mathbb{R})$ denote the space of bounded real-valued functions on J. We wish to prove the existence of a solution of the FIDE. Equation (1) in the space $AC(J,\mathbb{R})$ of all absolutely continuous real-valued functions on J.

We define a norm $\|.\|$ in $AC(J,\mathbb{R})$ by $\|x\|\sup_{t\in J}|x(t)|$. Clearly, $AC(J,\mathbb{R})$ becomes Banach algebra with this norm. The following definition is needed in the sequel.

Definition 3.1: A mapping $\beta : I\times C\to\mathbb{R}$ is Caratheodory if:

(i) $t\to\beta(t,x)$ is measurable for each $x\in C$, and
(ii) $x\to\beta(t,x)$ is continuous almost everywhere for $t\in I$.

Furthermore, a Caratheodory function β is L^1-Caratheodory if:

(iii) for each real number $r>0$ there exists a function $h_r\in L^1(I,\mathbb{R})$ such that $|\beta(t,x)|\le h_r(t)$, a.e. $t\in I$, for all $x\in C$ with $\|x\|_C\le r$.

Finally, a Caratheodory function β is $L^1{}_X$-Caratheodory if:

(iv) there exists a function $h\in L^1(I,\mathbb{R})$ such that $|\beta(t,x)|\le h(t)$, a.e. $t\in I$, for all $x\in C$.

For convenience, the function h is referred to as a bound function for $t\in I$, for all $x\in C$ with β. We will need the following hypotheses in the sequel.

(H_1) The function $f: I\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded with bound $F=\sup_{(t,x)\in I\times\mathbb{R}}|f(t,x)|$ there exists a bounded function $k: I\to\mathbb{R}$ with bound K satisfying $|f(t,x)-f(t,y)|\le k(t)|x-y|$, a.e. $t\in I$, for all $x,y\in\mathbb{R}$.

(H_2) The function $g(t,x)$ is $L^1{}_X$-Caratheodory with bound function h.

(H_3) There exists a continuous and nondecreasing function $\Omega:[0,\infty)\to(0,\infty)$ and a function $\gamma\in L^1(I,\mathbb{R})$ such that $\gamma(t)>0$, a.e. $t\in I$, and $|g(t,x)|\le\gamma(t)\Omega(\|x\|_C)$, a.e. $t\in I$, for all $x\in C$.

(B_1) The function $a: J\to J$ defined by the formulas $a(t)=h(t)t^{\alpha+1}$ is bounded on J and vanish at infinity, that is, $\lim_{t\to\infty}a(t)=0$.

Remark 3:1: Note that if hypotheses (B_1) hold. Then there exist constants $K_1 > 0$ and $K_2 > 0$ such that

$$K_1 = \sup\{\phi(t) : t \in J\}, \quad K_2 = \sup\left\{\frac{a(t)}{\Gamma(\alpha+2)} : t \in J\right\}$$

Lemma 3.1: If $h \in L^1(I, \mathbb{R}),$ then x is a solution of the FIDE

$$\left.\begin{aligned} D^{a}\left(\frac{x(t)}{f(t,x(t))}\right) &= \int_0^t h(s), \quad a.e. \quad t \in I, \\ x(t) &= \phi(t), \quad t \in I_0, \end{aligned}\right\} \tag{7}$$

if and only if it is solutions of the integral equation

$$x(t) = \left.\begin{cases} \left[f\left(t, x(t)\right)\right]\left(\phi(0) + \dfrac{1}{\Gamma\left(\alpha+1\right)}\displaystyle\int_0^t \left(t-s\right)^{\alpha} h\left(s\right) ds\right), & \text{if} \quad t \in I, \\ \phi(t), & \text{if} \quad t \in I_0 \end{cases}\right\} \tag{8}$$

The proof is immediately by applying $D^{-\alpha}$ on the two sides of the considered FIDE, then changing the order of the resultant double integral.

Proof. Integrating (7) of fractional order α, we get

$$I^{\alpha} D^{\alpha}\left(\frac{x(s)}{f\left(s, x(s)\right)}\right) ds = I^{\alpha}\left(\int_0^s h(\tau) d\tau\right)$$

$$\left(\frac{x(s)}{f\left(s, x(s)\right)}\right)_0^t = I^{\alpha}\left(\int_0^s h(\tau) d\tau\right)$$

$$\left(\frac{x(t)}{f\left(t, x(t)\right)}\right) = \left(\frac{x(0)}{f\left(0, x(0)\right)}\right) + I^{\alpha}\left(\int_0^s h(\tau) d\tau\right)$$

$$\left(\frac{x(t)}{f(t,x(t))}\right)=\phi(0)+I^{\alpha}\left(\int_0^s h(\tau)d\tau\right)$$

$$x(t)=f(t,x(t))\left[\phi(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^s h(\tau)d\tau\right)ds\right]$$

Since $\int_o^t f(t)dt^n=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}f(s)ds, n=0,1,2,.....$

The desired integral equation (8) follows by letting $n=2,$ we get

$$x(t)=\begin{cases}\left[f(t,x(t))\right]\left(\phi(0)+\dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^t(t-s)^{\alpha}h(s)ds\right), & \text{if } \quad t\in I,\\ \phi(t), & \text{if } \quad t\in I_0\end{cases}$$

Conversely, differentiated (8) of order α w.r.t.t we get

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)=D^{\alpha}(\phi(0))+D^{\alpha}I^{\alpha+1}h(t)$$

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)=0+I^{1}h(t)$$

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)=0+\int_0^t h(s)ds=\int_0^t h(s)ds,\ t\in I$$

Since, $\int_a^x (x-\eta)^{n-1}f(\eta)d\eta=\int_a^x f(\eta)d\eta$ (see Shanti (2006)).

Theorem 3.1: Assume that conditions $(H_1)-(H_3)$ and (B_1) hold. Further if $K(K_1+K_2)<1,$ where K_1 and K_2 are defined Remark (3.1). Then the FIDE (1) has a solution in the space $AC(J,\mathbb{R})$. Moreover, solutions of the equation (1) are locally attractive on J.

Proof: By a solution of the FIDE (1) we mean a continuous function $x:J\rightarrow\mathbb{R}$ that satisfies FIDE (1) on J.

Let $X = AC(J,\mathbb{R})$ be Banach Algebra of all absolutely continuous real valued function on J with the norm

$$\|x\| = \sup_{t\in J}|x(t)| \tag{9}$$

We shall obtain the solution of FIDE (1) under some suitable conditions on the functions involved in (1).

Now the FIDE (1) is equivalent to the functional integral equation (FIE)

$$x(t) = \begin{cases} \left[f\left(t,x(t)\right)\right]\left(\phi(0)+\dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^t (t-s)^{\alpha} g\left(s,x_s\right)ds\right), & \text{if } t\in I, \\ \phi(t), & \text{if } t\in I_0 \end{cases} \tag{10}$$

Consider the closed ball $\overline{B_r(0)}$ in X centered at origin 0 and of radius r, where $r = K(K_1+K_2) > 0$. Let us defined the two mappings $A: X \to X$ and $B:\overline{B_r(0)} \to X$ by

$$Ax(t) = \begin{cases} f(t,x(t)), & \text{if } t\in I, \\ 1, & \text{if } t\in I_0, \end{cases} \tag{11}$$

and
$$Bx(t) = \begin{cases} \phi(0)+\dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^t (t-s)^{\alpha} g(s,x_s)ds, & \text{if } t\in I \\ \phi(0), & \text{if } t\in I_0. \end{cases} \tag{12}$$

Thus from the FIDE (1) we obtain the operator equation as follows:

$$x(t) = Ax(t)Bx(t), \quad t\in J. \tag{13}$$

If operators A and B satisfy all the hypotheses of Theorem 2.1, then the operator equation (13) has solution on $\overline{B_r(0)}$.

Step 1: Firstly, we show that A is Lipschitz on $\overline{B_r(0)}$. Let $x, y \in \overline{B_r(0)}$; then by (H_1),

$$|Ax(t) - Ay(t)| \le |f(t, x(t)) - f(t, y(t))| \le k(t)|x(t) - y(t)| \le k(t)\|x - y\|$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|Ax - Ay\| \le \|k\|\|x - y\| \le K\|x - y\| \tag{14}$$

for all $x, y \in \overline{B_r(0)}$.

Thus, A is Lipschitz on $\overline{B_r(0)}$ with Lipschitz constant K.

Step 2: Secondly, we show that B is completely continuous operator on $\overline{B_r(0)}$. Using standard arguments such as those in Dugunji and Granas *et al.* (1982), it is can be shown that B is a continuous operator on $\overline{B_r(0)}$. We will show that $B\left(\overline{B_r(0)}\right)$ is a uniformly bounded and equicontinous set in $\overline{B_r(0)}$. Since $g(t, x_t)$ is $L^1{}_X$ -Caratheodory, we have

$$|Bx(t)| \le |\phi(0)| + \left|\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-x)^{\alpha} g(s, x_s)ds\right|$$

$$|Bx(t)| \le \|\phi\|_C + \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} |g(s, x_s)|ds$$

$$|Bx(t)| \le \|\phi\|_C + \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} \gamma(s)\Omega\left(\|x\|_C\right)ds$$

$$|Bx(t)| \le \|\phi\|_C + \frac{\sup_{s\in[0,T]}\gamma(s)}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} \Omega(r)ds$$

$$|Bx(t)| \le \|\phi\|_C + \frac{\sup_{s\in[0,T]}\gamma(s)\Omega(r)}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha\, ds. \tag{15}$$

Taking the supremum over t, we obtain $\|Bx\| \le M$ for all $x \in \overline{B_r(0)}$, where $M = \|\phi\|_C + \dfrac{\sup_{s\in[0,T]}\gamma(s)}{\Gamma(\alpha+2)}\Omega(r)T^{\alpha+1}.$

This shows that $B\left(\overline{B_r(0)}\right)$ is a uniformly bounded set in $\overline{B_r(0)}$.

To show that $B\left(\overline{B_r(0)}\right)$ is an equicontinuous set, let $t,\tau \in I$ be arbitrary. Then, for any $x \in \overline{B_r(0)}$, implies

$$|Bx(t_2) - Bx(t_1)| \le \left|\frac{1}{\Gamma(\alpha+1)}\int_0^{t_2}(t_2-s)^\alpha g(s,x_s)ds - \frac{1}{\Gamma(\alpha+1)}\int_0^{t_1}(t_1-s)^\alpha g(s,x_s)ds\right|$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[\int_0^{t_1}(t_2-s)^\alpha |g(s,x_s)|ds + \int_{t_1}^{t_2}(t_2-s)^\alpha |g(s,x_s)|ds - \int_0^{t_1}(t_1-s)^\alpha |g(s,x_s)|ds\right]$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[\int_0^{t_1}(t_2-s)^\alpha h(t_1)ds + \int_{t_1}^{t_2}(t_2-s)^\alpha h(t_1)ds - \int_0^{t_1}(t_1-s)^\alpha h(t_1)ds\right]$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[h(t_1)\int_0^{t_1}(t_2-s)^\alpha ds + h(t_2)ds + \int_{t_1}^{t_2}(t_2-s)^\alpha ds - h(t_1)\int_0^{t_1}(t_1-s)^\alpha ds\right]$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[\frac{h(t_1)}{\alpha+1}\int_0^{t_1}\left(t_2^{\alpha+1} - (t_2-t_1)^{\alpha+1}\right) + h(t_2)\frac{(t_2-t_1)^{\alpha+1}}{\alpha+1} - h(t_1)\frac{t_1^{\alpha+1}}{\alpha+1}\right]$$

$$\le \frac{1}{\Gamma(\alpha+2)}\left[h(t_1)\left(t_2^{\alpha+1} - t_1^{\alpha+1} - (t_2-t_1)^{\alpha+1}\right) + h(t_2)(t_2-t_1)^{\alpha+1}\right] \tag{16}$$

Because the right hand side of the inequality doesn't depends on x and tends to zero.

Therefore, $|Bx(t)-Bx(\tau)| \to 0$ as $t_1 \to t_2$.
Now, let $t_1 \in I_0$ and $t_2 \in I$. Then,

$$|Bx(t_2)-Bx(t_1)| \le \left| \phi(0) + \frac{1}{\Gamma(\alpha+1)} \int_0^{t_2} (t_2-s)^{\alpha} g(s,x_s)ds - \phi(t_1) \right|$$

$$\le |\phi(0)-\phi(t_1)| + \frac{T}{\Gamma(\alpha+1)} \left| \int_0^{t_2} g(s,x_s)ds \right|$$

$$\le |\phi(0)-\phi(t_1)| + |p(t_2)-p(0)|, \qquad (17)$$

where $p(t) = \frac{T}{\Gamma(\alpha+1)} \int_0^t h(s)ds.$

Note that if $t_1 \to t_2$, then $0 \to t_2$ and $t_1 \to 0$. Therefore from the above obtained estimates, it follows that:

$|\phi(0)-\phi(t_1)| \to 0, |p(t_2)-p(0)| \to 0;$ as $t_1 \to t_2$.

As a result, $|Bx_n(t_2)-Bx_n(t_1)| \to 0$ as $t_1 \to t_2$.
Similarly, if $\tau,\ t \in I_0$ then, $|Bx(t_1)-Bx(t_2)| \le |\phi(t_1)-\phi(t_2)|$.
Therefore, in all three cases, we see that
$|Bx(t_1)-Bx(t_2)| \to 0$ as $t_1 \to t_2$ for all $t_1, t_2 \in J$.

Hence, $B\left(\overline{B_r(0)}\right)$ is an equicontinuous set and so $B\left(\overline{B_r(0)}\right)$ is relatively compact by the Arzela-Ascoli Theorem. As a consequence, B is a compact and continuous operator on $\overline{B_r(0)}$.

Thus, B is a completely continuous on $\overline{B_r(0)}$.

Step 3: Also we have

$$M = \left\| B\left(\overline{B_r(0)}\right) \right\| = \sup\left\{ \|Bx\| : x \in \overline{B_r(0)} \right\}$$

$$\sup\left\{\sup_{t\in J}\left\{|\phi(0)|+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,|g(s,x_s)|\,ds\right\}:x\in\overline{B_r(0)}\right\}$$

$$\sup\left\{\sup_{t\in J}\left\{|\phi(0)|+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,h(s)ds\right\}:x\in\overline{B_r(0)}\right\}$$

$$\leq \sup_{t\in J}|\phi(0)|+\sup_{t\in J}\left[\frac{h(t)t^{\alpha+1}}{\Gamma(\alpha+2)}\right]$$

$$\leq \sup_{t\in J}|\phi(0)|+\sup_{t\in J}\left[\frac{a(t)}{\Gamma(\alpha+2)}\right]$$

$$\leq K_1+K_2 \tag{18}$$

and therefore $MK = K(K_1+K_2)<1.$

Thus, the condition (d) of Theorem 2.1 are satisfied.
Hence, all the conditions of Theorem 2.1 are satisfied and therefore the operator equation $AxBx = x$ has a solution in $\overline{B_r(0)}$. As a result, the FIDE (1) has a solution defined on J.

Step 4: Finally, we show the local attractively of the solutions for FIDE (1). Let x and y be any two solutions of the FIDE (1) in $\overline{B_r(0)}$ defined on J. Then, we have

$$\begin{aligned}&|x(t)-y(t)|\\ &\leq \left|f(t,x(t))\left(\phi(0)+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,g(s,x_s)ds\right)\right|\\ &+\left|f(t,y(t))\left(\phi(0)+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}\,g(s,y_s)ds\right)\right|\end{aligned}$$

$$\leq F\left(|\phi(0)|+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} h(t)ds\right)$$

$$+F\left(|\phi(0)|+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} h(t)ds\right)$$

$$\leq 2F\left(|\phi(0)|+\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha} h(t)ds\right)$$

$$\leq 2F\left(|\phi(0)|+\frac{h(t)t^{\alpha+1}}{\Gamma(\alpha+2)}\right)$$

$$\leq 2F\left(|\phi(0)|+\frac{a(t)}{\Gamma(\alpha+2)}\right) \tag{19}$$

for all $t \in J$.

Since $\lim_{t\to\infty} a(t)=0$, for $\varepsilon>0$, there is real numbers $T>0$ such that $a(t)\leq \dfrac{\Gamma(\alpha+2)\varepsilon}{2F}$ for all $t\geq T$ and if $|\phi(0)|=0$ then from the above inequality it follows that $|x(t)-y(t)|\leq\varepsilon$ for all $t\geq T$. This completes the proof.

4. CONCLUSION

In this paper, we have presented a solution of a generalized fractional order functional integro-differential equation with the help of the Dhage's Hybrid fixed point theorem. It is expected that result derived in this survey may find applications in the solution of certain fractional order differential and integral equations arising problems of physical sciences and engineering areas.

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